Pre-Michaelmas Maths Problem Set

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You will have come from diverse academic backgrounds, each emphasizing different areas of mathematics. The purpose of this course is to ensure that each of you is fluent in the maths considered fundamental to physics, to introduce you to university style teaching and to prepare you for some of the more demanding courses that you will encounter in your first year. The perfect solution of every problem is neither required nor expected; endeavour to solve each problem to the best of your ability, providing reasoning wherever you become stuck.

I. DIFFERENTIATION

1. Given the definition of the derivative as

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{y(x+\delta x) - y(x)}{\delta x} \right),$$

evaluate:

   (a) $\dfrac{d}{dx}\left(x^2\right)$

   (b) $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$

2. Find the derivative of the following functions:

   (a) $x^2 \log_e x$

   (b) $(\sin x - 2\cos x)^2$

   (c) $\left(1 - x^{\frac{1}{2}}\right)\left(1 + x^{\frac{1}{3}}\right)$

   (d) e^{2x}

   (e) $x^3 f(x)$, where $f(x)$ is an arbitrary function of x.

3. Use the quotient rule (or product rule) to differentiate:

   (a) $\tan x$

   (b) $\cot x$

4. Differentiate the following functions:

   (a) $x^2 \sin x + \log_e x$

   (b) $\left(x^3 - 6x\right)\left(2x^2 + 5x - 1\right)$

   (c) $\dfrac{1}{\sin x}$

   (d) $\dfrac{x^2 + 4x - 1}{x^2 + 9}$

   (e) $\dfrac{x \tan x}{9x^2 - 2x + 1}$

   (f) $2^x + x^2 + e^2$

   (g) $\left(e^x + e^{-x}\right)^2$

5. If the displacement of a particle $x(t)$ is given by the function:

$$x(t) = e^{-\alpha t}\left(A\cos(\omega t) + B\sin(\omega t)\right),$$

where α, ω, A, and B are constants, find expressions for the velocity and acceleration of the particle, and show that:

$$\ddot{x}(t) + 2\alpha\dot{x}(t) + \left(\omega^2 + \alpha^2\right)x(t) = 0.$$

II. INTEGRATION

1. Find the integral $\int x^2 \cos\left(x^3\right) dx$, using the substitution $y = x^3$.

2. Find the integral $\int \cos x e^{\sin x} dx$, using the substitution $y = \sin x$.

3. Find the integral $\int \sin x \,(2 + \cos x)^7 \, dx$

4. Find the integral $\int \dfrac{x}{x^2+1} dx$.

5. Find $\int x(1+x)^{\frac{5}{2}}$, using the substitution $y = 1 + x$.

6. Consider $\int \cos^4 x dx = \int f(x)g(x)dx$, with $f(x) = \cos x$ and $g(x) = \cos^3 x$, and show that:

$$\int \cos^4 x dx = \frac{1}{4}\sin x \cos^3 x + \frac{3}{4}\int \cos^2 x dx.$$

7. Find the integral $\int \dfrac{dx}{\sqrt{1-16x^2}}$.

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8. Find the integral $\int \frac{dx}{5+4x^2}$.

9. Find the integral $\int \frac{1}{9-4x^2}dx$, using partial fractions.

10. Find the integral $\int \frac{dx}{3x^2-6x+7}$, by first completing the square.

11. Evaluate the definite integral $\int_3^7 \frac{dx}{x^2-6x+25}$.

III. SERIES EXPANSION

Verify all of the following series expansions:

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

2. $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

3. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

4. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

5. $(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$

IV. GRAPH SKETCHING

1. Determine whether the following functions are even, odd or neither:

   (a) $x + \sin x$
   (b) $x^2 - 2\cos x$
   (c) $x^2 + \sin x$
   (d) $x \sin x$
   (e) $x \cos x$

   Sketch the following functions:

   (a) $f(x) = x - \frac{1}{x}$
   (b) $f(x) = x^4 - 4x^3 + 10$
   (c) $f(x) = x^4 - x^2$

V. COMPLEX NUMBERS

1. For (i) $z_1 = 1+i$, $z_2 = -3+2i$, and (ii) $z_1 = 2e^{i\pi/4}$, $z_2 = e^{-3i\pi/4}$, find:

   (a) $z_1 + z_2$
   (b) $z_1 - z_2$
   (c) $z_1 z_2$
   (d) $\frac{z_1}{z_2}$
   (e) $|z_1|$
   (f) z_1^*

2. Write the following complex numbers in the form $a + ib$, where a and b are real:

   (a) e^i
   (b) $\sqrt{i}$
   (c) $\ln i$
   (d) $\cos i$
   (e) $\sin i$
   (f) $\sinh(x+iy)$
   (g) $\ln\left(\frac{1}{2}(\sqrt{3}+i)\right)$
   (h) $(1+i)^{iy}$

   (note that $i = e^{i\pi/2}$).

3. Write the following in polar form ($z = re^{i\theta}$)

   (a) $-i$
   (b) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$
   (c) $1+i$
   (d) $1-i$
   (e) $\frac{1+i}{1-i}$

4. Sketch on separate Argand diagrams the lines and areas represented by:

   (a) $|z| < 1$
   (b) $\mathrm{Re}(z) = 2$
   (c) $-\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{4}$
   (d) $|z-1| = |z+i|$
   (e) $z = te^{it}$ for $0 \leq t \leq 2\pi$

5. Plot the following complex numbers on the same Argand diagram: i, $\frac{1+i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$, 1, $-i$, $\frac{-1-i}{\sqrt{2}}$, -1, $\frac{-1+i}{\sqrt{2}}$. How are the points related?

6. By noting that $e^{i5\theta} = (\cos\theta + i\sin\theta)^5$, express $\cos 5\theta$ as a polynomial in $\cos\theta$.

7. Show that: $\sum_{n=0}^{\infty} 2^{-n}\cos n\theta = \frac{1 - 1/2\cos\theta}{5/4 - \cos\theta}$

VI. VECTORS

1. If $\underline{a} = \left[3, 7, 5\right]$ and $\underline{b} = [8, 1, 3]$, find $3\underline{a} - 2\underline{b}$.
2. If $\underline{u} = 8\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$ and $\underline{v} = 9\hat{\imath} + 5\hat{\jmath} + \hat{k}$, find $2\underline{u} + 3\underline{v}$.
3. What is the norm of the vector $[3, 8, 3, 3, 3]$?
4. If $[1/6, 1/4, x, 1/3]$ is a unit vector, what is the value of x?
5. If $\underline{y} = [3, 7, 1, 3]$, find $\|\underline{y}\|\, \underline{y}$.
6. Find $[1, 6, 3, 8] \cdot [4, 2, 6, 1]$.
7. If $\underline{a} = [1, 4, 3]$, $\underline{b} = [2, 4, 1]$, and $\underline{c} = [7, 1, 0]$, find $\underline{a}\,(\underline{b} \cdot \underline{c})$.
8. Find the angle between the vectors $[2, 2, 1]$ and $[2, 3, 6]$.
9. What is the vector product of $[2, 3, 4]$ and $[1, 3, 5]$?
10. Find $[5, 6, 7] \cdot ([2, 2, 7] \times [3, 7, 8])$.
11. Consider the vector equation of a straight line: $\underline{r} = \underline{a} + \lambda \underline{b}$. $\underline{r}$ is the position vector for a point on the line and λ is a free (scalar) parameter.
    (a) What do the vectors $\underline{a}$ and $\underline{b}$ represent?
    (b) If the Cartesian components of $\underline{r}$ are $[x, y, z]$, those of $\underline{a}$ are $[a_x, a_y, a_z]$, and those of $\underline{b}$ are $[b_x, b_y, b_z]$, show that:
    $$\frac{x - a_x}{b_x} = \frac{y - a_y}{b_y} = \frac{z - a_z}{b_z}.$$
    This is the Cartesian equation of a line.
    (c) Write the equation of the line:
    $$x = \frac{y}{2} = \frac{z - 1}{3}$$
    in vector form.
12. Which of the following sets of vectors are linearly independent? For each set that is not, identify a subset of linearly independent vectors.
    (a) $\underline{v_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \underline{v_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \underline{v_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$
    (b) $\underline{v_1} = \begin{bmatrix}0\\1\\1\end{bmatrix}, \underline{v_2} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \underline{v_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$
    (c) $\underline{v_1} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \underline{v_2} = \begin{bmatrix}2\\3\\1\end{bmatrix}, \underline{v_3} = \begin{bmatrix}1\\6\\-1\end{bmatrix}$
    (d) $\underline{v_1} = \begin{bmatrix}1\\2\\0\\-3\end{bmatrix}, \underline{v_2} = \begin{bmatrix}2\\1\\1\\-4\end{bmatrix}, \underline{v_3} = \begin{bmatrix}-3\\6\\-4\\1\end{bmatrix}$
13. What are the dimensions of the vector spaces spanned by the following sets of vectors?
    (a) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$
    (b) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$
    (c) $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}$
    (d) $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$
    (e) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\0\\2\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$

    If the number of vectors is greater than the dimension, choose a subset to form a basis, and express the remaining vector(s) in that basis.
    Which of the bases are orthogonal?

VII. MATRICES

1. Consider the following matrices, **A**, **B**, **C**, **D**. Where it makes sense to do so, calculate their sums and products. $\mathbf{A} = \begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}, \mathbf{B} = \begin{bmatrix}0 & 1\\3 & 2\\1 & 0\end{bmatrix}$, $\mathbf{C} = \begin{bmatrix}-1 & 2 & 3\\-2 & 1 & 0\end{bmatrix}, \mathbf{D} = \begin{bmatrix}-1 & 12\\6 & 0\end{bmatrix}$
2. For $\mathbf{A} = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\end{bmatrix}, \mathbf{B} = \begin{bmatrix}-1 & 0\\0 & 3\\-1 & 0\end{bmatrix}, \mathbf{C} = \begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}$, calculate the products **AB**, **BA**, **CA**, **BC**.
3. For $\mathbf{A} = \begin{bmatrix}1 & 3 & 0\\2 & 1 & 1\end{bmatrix}, \mathbf{B} = \begin{bmatrix}1 & 0\\2 & 1\\-1 & -1\end{bmatrix}, \mathbf{C} = \begin{bmatrix}2 & 1\\-1 & 1\\0 & 1\end{bmatrix}$, verify the distributive law $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.
4. If $\mathbf{A} = \begin{bmatrix}4 & 2\\2 & 1\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}-2 & -1\\4 & 2\end{bmatrix}$, show that $\mathbf{AB} = \mathbf{0}$, but $\mathbf{BA} \neq \mathbf{0}$.
5. If $\mathbf{A} = \begin{bmatrix}2 & 3\\4 & 1\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}-4 & 2\\3 & 1\end{bmatrix}$,
    (a) Find $\det(\mathbf{A})$ and $\det(\mathbf{B})$.
    (b) Calculate $\det(\mathbf{AB})$.

    You should find that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ – this is true generally.

6. Show that the space of 2×2 matrices is a linear vector space.
   (a) What is its dimension?
   (b) Give a basis for the space.
7. What is the dimension of the space of $n \times n$ matrices? Describe a basis for this space.
8. What is the dimension of the space of diagonal $n \times n$ matrices?
9. What is the dimension of the space of symmetric 2×2 matrices?

   A symmetric matrix has $\mathbf{A}^\intercal = \mathbf{A}$, where $A^\intercal_{ij} = A_{ji}$ (the first index labels the row and the second the column for each matrix element).

   Give a basis for this space.
10. Consider two 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$. Show explicitly that:

    $$(\mathbf{AB})_{23} = \sum_{j=1}^{3} A_{2j} B_{j3}.$$

    More generally, matrix multiplication can be defined by the relation:

    $$(\mathbf{AB})_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}.$$
11. Show that for general $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$:

    $(\mathbf{AB})^\intercal = \mathbf{B}^\intercal \mathbf{A}^\intercal$.

    Hint: consider the matrix element $(\mathbf{AB})^\intercal_{ij} = (AB)_{ji}$, and use the definition of matrix multiplication given above.
12. In Cartesian co-ordinates, a position vector $\underline{r}$ can be written:

    $$\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

    In a co-ordinate system rotated by an angle θ about the z-axis the vector can be written as:

    $$\underline{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

    where $x' = x\cos\theta + y\sin\theta$, $y' = -x\sin\theta + y\cos\theta$ and $z' = z$. Show that this transformation can be written as $\underline{r}' = \mathbf{R}_z\underline{r}$, where the matrix $\mathbf{R}_z$ is given by:

    $$\mathbf{R}_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

    The length of the vector is unchanged if $x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$. Show that this condition can be written $\underline{r}'^\intercal \underline{r}' = \underline{r}^\intercal \underline{r}$.

    If $\underline{r}'$ and $\underline{r}$ are related by a general rotation $\mathbf{R}$, such that $\underline{r}' = \mathbf{R}\underline{r}$, show that the condition $\underline{r}'^\intercal \underline{r}' = \underline{r}^\intercal \underline{r}$ corresponds to $\mathbf{R}^\intercal \mathbf{R} = \mathbf{I}$.

    A matrix satisfying this condition is known as 'orthogonal'.
13. Calculate the determinants of the matrices:

    $$\mathbf{A} = \begin{bmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{bmatrix} \quad \mathbf{B} = \frac{1}{\sqrt{8}} \begin{bmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{bmatrix}$$

    Are the matrices:
    (a) real,
    (b) diagonal,
    (c) symmetric,
    (d) antisymmetric,
    (e) singular,
    (f) orthogonal,
    (g) Hermitian,
    (h) anti-Hermitian,
    (i) and/or unitary?
14. Write the following simultaneous equations as a matrix equation $\mathbf{A}\underline{x} = \underline{0}$, where $\mathbf{A}$ is a square matrix and $\underline{x}$ is a column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

    $$\begin{aligned} 2x + 3z &= 0 \\ 4x + 2y + 5z &= 0 \\ x - y + 2z &= 0 \end{aligned}$$

    $\underline{x} = \underline{0}$ is known as the 'trivial' solution. What condition must be satisfied by $\det(\mathbf{A})$ for the equation $\mathbf{A}\underline{x} = \underline{0}$ to have a non-trivial solution?

    Show that that condition is satisfied in this case, and find a solution in the form of the vector equation for a line.
15. The matrix equation $\mathbf{A}\underline{x} = \lambda\underline{x}$, where λ is a scalar, is often known as the 'eigenvalue problem'.

    Vectors $\underline{x}$ that satisfy the equation are known as eigenvectors of $\mathbf{A}$, and the corresponding values of λ are known as eigenvalues.

    Rewriting the equation as $(\mathbf{A} - \lambda\mathbf{I})\,\underline{x} = \underline{0}$, one can see that non-trivial solutions for $\underline{x}$ may be found if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Using this condition, find the eigenvalues of the matrix:

    $$\begin{bmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

    Find an eigenvector corresponding to the eigenvalue 4.

VIII. ORDINARY DIFFERENTIAL EQUATIONS

1. Solve the first order ODE's:

   (a) $\dfrac{dy}{dx} = \dfrac{y+1}{x+1}$, given that $y = 1$ at $x = 0$.

   (b) $x\dfrac{dy}{dx} + 3y = 2$, given that $y = 2$ at $x = 1$.

   (c) $\dfrac{dy}{dx} + \dfrac{y}{x} = \sin x$, given that $y = 0$ at $x = \pi$.

2. Solve the second order ODE:

   $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$

   with boundary conditions $y = 2$, and $\dfrac{dy}{dx} = 1$ at $x = 0$.

3. Find the general solutions of the second order ODE's:

   (a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$.

   (b) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 2\cos^2 x$.

   (c) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$ (using the substitution $x = e^t$ or otherwise).