

PROBLEM SHEET 1

1.1 Find the radius and centre of the circle described by the equation

$$x^2 + y^2 - 2x - 4y + 1 = 0$$

by writing it in the form $(x - a)^2 + (y - b)^2 = c^2$ for suitable a, b and c .

1.2 Find the equation of the line perpendicular to $y = 3x$ passing through the point $(3, 9)$.

1.3 Given

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \quad \text{and} \quad \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

show that

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)] \quad \text{and} \quad \sin^2 A = \frac{1}{2}[1 - \cos 2A].$$

1.4 Show that

$$4 \cos(\alpha t) + 3 \sin(\alpha t) = 5 \cos(\alpha t + \phi)$$

where $\phi = \arctan(-3/4)$.

1.5 Show that, for $-1 \leq x \leq 1$,

$$\cos(\sin^{-1} x) = \pm \sqrt{1 - x^2}.$$

1.6 Given

$$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B \quad \text{and} \quad \cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B,$$

show that

$$\cosh A \cosh B = \frac{1}{2}[\cosh(A + B) + \cosh(A - B)] \quad \text{and} \quad \sinh^2 A = \frac{1}{2}[\cosh 2A - 1].$$

1.7 Given that

$$\sinh x = \frac{1}{2}[e^x - e^{-x}],$$

show that

$$\sinh^{-1} x = \ln \left[x + \sqrt{1 + x^2} \right].$$

1.8 Express

$$\frac{x}{(x - 1)(x - 2)}$$

in partial fractions.

1.9 If $a_n = \frac{1}{n}$, find $\sum_{i=1}^5 a_n$ as a fraction.

1.10 If $S = \sum_{i=0}^N x^i$, show that $xS = \sum_{i=1}^{N+1} x^i$. Hence show that $S - xS = 1 - x^{N+1}$ and therefore that

$$S = \frac{1 - x^{N+1}}{1 - x}.$$

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PROBLEM SHEET 2

2.1 Given that

$$\sinh x = \frac{1}{2}[e^x - e^{-x}]$$

show that

$$\frac{dy}{dx} = \cosh x.$$

2.2 Given that

$$\cosh x = \frac{1}{2}[e^x + e^{-x}],$$

show that

$$\frac{dy}{dx} = \sinh x.$$

2.3 Let n be a positive integer. Show that

$$\frac{d^n(x^n)}{dx^n} = n!$$

2.4 If $y = \ln x$, show that

$$\frac{dy}{dx} = \frac{1}{x}; \quad \frac{d^2y}{dx^2} = \frac{-1}{x^2}; \quad \frac{d^{100}y}{dx^{100}} = \frac{-99!}{x^{100}}.$$

2.5 Find the equation of the tangent to the curve $y = x^2$ at $(1, 1)$.

2.6 Find the slope of the curve $y = 4x + e^x$ at $(0, 1)$.

2.7 Find the angle of inclination of the tangent to the curve $y = x^2 + x + 1$ at the point $(0, 1)$.

2.8 The displacement $y(t)$ metres of a body at time t seconds ($t \geq 0$) is given by $y(t) = t - \sin t$. At what times is the body at rest?

2.9 A particle has displacement $y(t)$ metres at time t seconds given by $y(t) = 3t^3 + 4t + 1$. Find its acceleration at time $t = 4$ seconds.

2.10 If

$$y = \sum_{n=0}^N a_n x^n$$

show that

$$\frac{dy}{dx} = \sum_{n=1}^N n a_n x^{n-1}.$$

PROBLEM SHEET 3

3.1 If $y = \ln(1 + x^2)$, find dy/dx .

3.2 If

$$y = \frac{x}{1 + x^2}$$

find dy/dx .

3.3 If $y = \cosh(x^4)$, find dy/dx .

3.4 If $y = x^2 \ln x$, find d^2y/dx^2 .

3.5 Find dy/dx for $y = (1 + x^2)^{-1/2}$.

3.6 Show that for $y = \sinh^{-1} x$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}.$$

3.7 Show that for $y = \ln[x + \sqrt{1 + x^2}]$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}.$$

3.8 Find dy/dx for $y = \cos^{-1}(\sin x)$.

3.9 A curve is given in polar coordinates by $r = 1 + \sin^2 \theta$. Find dy/dx at $\theta = \pi/4$.

3.10 Show that if

$$y = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right|, \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{x^2 - a^2}.$$

PROBLEM SHEET 4

4.1 Given $f(x - ct)$, where x and c are constant, show that

$$\frac{d^2}{dt^2}f(x - ct) = c^2 f''(x - ct),$$

and calculate this expression when $f(u) = \sin u$.

4.2 Classify the stationary point of $y = x^{-2}\ln x$, where $x > 0$.

4.3 Classify the stationary points of $y(x) = x^2 - 3x + 2$.

4.4 The numbers x and y are subject to the constraint $x + y = \pi$. Find the values of x and y for which $\cos(x)\sin(y)$ takes its minimum value.

4.5 Sketch the graph of

$$y = \frac{x}{1 + x^2}.$$

4.6 Sketch the graph of

$$y(x) = \tan(2x) \quad \text{for} \quad -\frac{3\pi}{4} \leq x \leq \frac{3\pi}{4}.$$

4.7 Sketch the graph of $y = x \ln x$ for $x > 0$.

4.8 Sketch the graph of

$$y = \frac{x^3}{2x - 1}$$

showing clearly on your sketch any asymptotes.

4.9 Sketch the graph of

$$y = x \cos(3x) \quad \text{for} \quad 0 \leq x \leq 2\pi.$$

PROBLEM SHEET 5

5.1 Verify the following Taylor expansions (taking the ranges of validity for granted).

(a)

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots \text{valid for any } x.$$

(b)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots \text{valid for any } x.$$

(c)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots \text{valid for any } x.$$

(d) Let α be a constant.

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \text{valid for } -1 < x < 1$$

(e)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n-1} x^n}{n} + \dots \text{valid for } -1 < x \leq 1.$$

5.2 Obtain a four-term Taylor polynomial approximation valid near $x = 0$ for each of the following.

$$(a) \quad (1+x)^{1/2}, \quad (b) \quad \sin(2x), \quad (c) \quad \ln(1+3x).$$

PROBLEM SHEET 6

6.1 Reduce to standard form

$$(a) \frac{3+i}{4-i}, \quad \text{and} \quad (b) (1+i)^5.$$

6.2 Prove

$$(a) |z_1 z_2| = |z_1| |z_2|, \quad \text{and} \quad (b) \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad \text{when } z_2 \neq 0.$$

6.3 Given that $e^{i\theta} = \cos \theta + i \sin \theta$, prove that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

6.4 Let $z = 1+i$. Find the following complex numbers in standard form and plot their corresponding points in the Argand diagram:-

$$(a) \bar{z}^2, \quad \text{and} \quad (b) \frac{z}{\bar{z}}.$$

6.5 Find the modulus and principal arguments of (a) $-2+2i$, (b) $3+4i$.

6.6 Find all the complex roots of

- (a) $\cosh z = 1$;
- (b) $\sinh z = 1$;
- (c) $e^z = -1$;
- (d) $\cos z = \sqrt{2}$.

6.7 Show that the mapping

$$w = z + \frac{c}{z},$$

where $z = x + iy$, $w = u + iv$ and c is a real number, maps the circle $|z| = 1$ in the z plane into an ellipse in the w plane and find its equation.

6.8 Show that

$$\cos^6 \theta = \frac{1}{32}(\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10).$$

PROBLEM SHEET 7

7.1 The matrix $A = (a_{ij})$ is given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & -2 & 4 \\ 1 & 5 & -3 \end{pmatrix}$$

Identify the elements a_{13} and a_{31} .

7.2 Given that

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ -1 & 1 \\ -0 & 1 \end{pmatrix},$$

verify the distributive law $A(B + C) = AB + AC$ for the three matrices.

7.3 Let

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}.$$

Show that $AB = 0$, but that $BA \neq 0$.

7.4 A general $n \times n$ matrix is given by $A = (a_{ij})$. Show that $A + A^T$ is a symmetric matrix, and that $A - A^T$ is skew-symmetric.

Express the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

as the sum of a symmetric matrix and a skew-symmetric matrix.

7.5 Let the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 1 \end{pmatrix}.$$

Find A^2 . For what relation between a, b , and c is $A^2 = I$ (the unit matrix)? In this case, what is the inverse matrix of A ? What is the inverse matrix of A^{2n-1} (n a positive integer)?

7.6 Using the rule for inverses of 2×2 matrices, write down the inverse of

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

7.7 If A and B are both $n \times n$ matrices with A non-singular, show that

$$(A^{-1}BA)^2 = A^{-1}B^2A.$$

PROBLEM SHEET 8

8.1 Obtain the components of the vectors below where L is the magnitude and θ the angle made with the positive direction of the x axis ($-180^\circ < \theta \leq 180^\circ$).

- (a) $L = 3, \theta = 60^\circ$;
- (b) $L = 3, \theta = -150^\circ$.

8.2 Two ships, S_1 and S_2 set off from the same point Q . Each follows a route given by successive displacement vectors. In axes pointing east and north, S_1 follows the path to B via $\overrightarrow{QA} = (2, 4)$, and $\overrightarrow{AB} = (4, 1)$. S_2 goes to E via $\overrightarrow{QC} = (3, 3)$, $\overrightarrow{CD} = (1, 1)$ and $\overrightarrow{DE} = (2, -3)$. Find the displacement vector $\overrightarrow{BE}$ in component form.

8.3 Sketch a diagram to show that if A, B, C are any three points, then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$. Formulate a similar result for any number of points.

8.4 Sketch a diagram to show that if A, B, C, D are any four points, then $\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AD}$. Formulate a similar result for any number of points.

8.5 Two points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. In terms of $\mathbf{a}$ and $\mathbf{b}$ find the position vectors of the following points on the straight line passing through A and B .

- (a) The mid-point C of AB ;
- (b) a point U between A and B for which $AU/UB = 1/3$.

8.6 Suppose that C has position vector $\mathbf{r}$ and $\mathbf{r} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$ where λ is a parameter, and A, B are points with $\mathbf{a}, \mathbf{b}$ as position vectors. Show that C describes a straight line. Indicate on a diagram the relative positions of A, B, C , when $\lambda < 0$, $0 < \lambda < 1$, and $\lambda > 1$.

8.7 Find the shortest distance from the origin of the line given in vector parametric form by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where

$$\mathbf{a} = (1, 2, 3) \quad \text{and} \quad \mathbf{b} = (1, 1, 1),$$

and t is the parameter (Hint: use a calculus method, with t as the independent variable.)

8.8 $ABCD$ is any quadrilateral in three dimensions. Prove that if P, Q, R, S are the mid-points of AB, BC, CD, DA respectively, then $PQRS$ is a parallelogram.

8.9 ABC is a triangle, and P, Q, R are the mid-points of the respective sides BC, CA, AB . Prove that the medians AP, BQ, CR meet at a single point G (called the centroid of ABC ; it is the centre of mass of a uniform triangular plate.)

8.10 Show that the vectors $\overrightarrow{OA} = (1, 1, 2)$, $\overrightarrow{OB} = (1, 1, 1)$, and $\overrightarrow{OC} = (5, 5, 7)$ all lie in one plane.

PROBLEM SHEET 9

9.1 The figure $ABCD$ has vertices at $(0, 0)$, $(2, 0)$, $(3, 1)$ and $(1, 1)$.

Find the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$. Find $\overrightarrow{AC} \cdot \overrightarrow{BD}$.

Hence show that the angles between the diagonals of $ABCD$ have cosine $-1/\sqrt{5}$.

9.2 Show that the vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ are perpendicular.

Obtain any vector $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

9.3 Find the value of λ such that the vectors $(\lambda, 2, -1)$ and $(1, 1, -3\lambda)$ are perpendicular.

9.4 Find a constant vector parallel to the line given parametrically by

$$x = 1 - \lambda, y = 2 + 3\lambda, z = 1 + \lambda.$$

9.5 A circular cone has its vertex at the origin and its axis in the direction of the unit vector $\hat{\mathbf{a}}$. The half-angle at the vertex is α . Show that the position vector $\mathbf{r}$ of a general point on its surface satisfies the equation

$$\hat{\mathbf{a}} \cdot \mathbf{r} = |\mathbf{r}| \cos \alpha.$$

Obtain the cartesian equation when $\hat{\mathbf{a}} = (2/7, -3/7, -6/7)$ and $\alpha = 60^\circ$.

PROBLEM SHEET 10

10.1 For vectors $\mathbf{a}$ and $\mathbf{b}$, show

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \quad \text{and} \quad \mathbf{a} \cdot \mathbf{b} = \frac{1}{4}(|\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2)$$

10.2 In component form, let $\mathbf{a} = (1, -2, 2)$, $\mathbf{b} = (3, -1, -1)$, and $\mathbf{c} = (-1, 0, -1)$. Evaluate

$$\mathbf{a} \times \mathbf{b}, \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \quad \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

10.3 What is the geometrical significance of $\mathbf{a} \times \mathbf{b} = \mathbf{0}$?

10.4 Show that the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ are perpendicular. Find a vector which is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

10.5 Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors, and $\mathbf{v}$ be any vector. Show that $\mathbf{v}$ can be expressed as

$$\mathbf{v} = X\mathbf{a} + Y\mathbf{b} + Z\mathbf{c}$$

where X, Y, Z , are constants given by

$$X = \frac{\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \quad Y = \frac{\mathbf{v} \cdot (\mathbf{c} \times \mathbf{a})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \quad Z = \frac{\mathbf{v} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}.$$

(Hint: start by forming, say, $\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c})$).

PROBLEM SHEET 11

11.1 Integrate $\cos(3x + 4)$.

11.2 Integrate $(1 - 2x)^{10}$.

11.3 Integrate e^{4x-1} .

11.4 Integrate $(4x + 3)^{-1}$.

11.5 Find the equation of the curve passing through the point $(1, 2)$ satisfying $dy/dx = 2x$.

11.6 A particle has acceleration $(3t^2 + 4) \text{ ms}^{-2}$ at time t seconds. If its initial speed is 5ms^{-1} , what is its speed at time $t = 2$ seconds?

11.7 Find the area between the graph of $y = \sin x$ and the x -axis from $x = 0$ to $x = \pi/2$.

11.8 Find the area between the graph

$$y = \frac{1}{x-1}$$

and the x -axis between $x = 2$ and $x = 3$.

11.9 Find the signed area between the graph $y = 2x + 1$ and the x -axis between $x = -1$ and $x = 3$.

11.10 Find y , given that

$$\frac{d^2y}{dx^2} = \sin x - \frac{4}{x^3}.$$