

INDUCTION EXERCISES 1

1. Factorials are defined inductively by the rule

$$0! = 1 \quad \text{and} \quad (n+1)! = n! \times (n+1).$$

Then binomial coefficients are defined for $0 \leq k \leq n$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Prove from these definitions that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1},$$

and deduce the Binomial Theorem: that for any x and y ,

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

2. Prove that

$$\sum_{r=1}^n \frac{1}{r^2} \leq 2 - \frac{1}{n}.$$

3. Prove that for $n = 1, 2, 3, \dots$

$$\sqrt{n} \leq \sum_{k=1}^n \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1.$$

4. Let $A = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$. Show that

$$A^n = 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}$$

for $n = 1, 2, 3, \dots$. Can you find a matrix B such that $B^2 = A$?

5. Let k be a positive integer. Prove by induction on n that

$$\sum_{r=1}^n r(r+1)(r+2) \cdots (r+k-1) = \frac{n(n+1)(n+2) \cdots (n+k)}{k+1}.$$

Show now by induction on k that

$$\sum_{r=1}^n r^k = \frac{n^{k+1}}{k+1} + E_k(n)$$

where $E_k(n)$ is a polynomial in n of degree at most k .

INDUCTION EXERCISES 2.

1. Show that n lines in the plane, no two of which are parallel and no three meeting in a point, divide the plane into

$$\frac{n^2 + n + 2}{2}$$

regions.

2. Prove for every positive integer n , that

$$3^{3n-2} + 2^{3n+1}$$

is divisible by 19.

3. (a) Show that if $u^2 - 2v^2 = 1$ then

$$(3u + 4v)^2 - 2(2u + 3v)^2 = 1.$$

(b) Beginning with $u_0 = 3, v_0 = 2$, show that the recursion

$$u_{n+1} = 3u_n + 4v_n \quad \text{and} \quad v_{n+1} = 2u_n + 3v_n$$

generates infinitely many integer pairs (u, v) which satisfy $u^2 - 2v^2 = 1$.

(c) How can this process be used to produce better and better rational approximations to $\sqrt{2}$? How many times need this process be repeated to produce a rational approximation accurate to 6 decimal places?

4. The Fibonacci numbers F_n are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2$$

and $F_0 = 0$ and $F_1 = 1$. Prove for every integer $n \geq 0$, that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

[Hint: you may find it helpful to show first that the two roots of the equation $x^2 = x + 1$ are α and β .]

5. The sequence of numbers x_0, x_1, x_2, \dots begins with $x_0 = 1$ and $x_1 = 1$ and is then recursively determined by the equations

$$x_{n+2} = 4x_{n+1} - 3x_n + 3^n \quad \text{for } n \geq 0.$$

(a) Find the values of x_2, x_3, x_4 and x_5 .

(b) Can you find a solution of the form

$$x_n = A + B \times 3^n + C \times n3^n$$

which agrees with the values of x_0, \dots, x_5 that you have found?

(c) Use induction to prove that this is the correct formula for x_n for all $n \geq 0$.

ALGEBRA EXERCISES 1

1. (a) Find the remainder when $n^2 + 4$ is divided by 7 for $0 \leq n < 7$.

Deduce that $n^2 + 4$ is not divisible by 7, for every positive integer n . [Hint: write $n = 7k + r$ where $0 \leq r < 7$.]

(b) Now k is an integer such that $n^3 + k$ is not divisible by 4 for all integers n . What are the possible values of k ?

2. (i) Prove that if a, b are positive real numbers then

$$\sqrt{ab} \leq \frac{1}{2}(a + b).$$

(ii) Now let a_1, a_2, \dots, a_n be positive real numbers. Let $S = a_1 + a_2 + \dots + a_n$ and $P = a_1 a_2 \dots a_n$.

Suppose that a_i and a_j are distinct. Show that replacing a_i and a_j with $(a_i + a_j)/2$ and $(a_i + a_j)/2$ increases P without changing S .

Deduce that

$$(a_1 a_2 \dots a_n)^{1/n} \leq \frac{a_1 + a_2 + \dots + a_n}{n}.$$

3. (i) Let n be a positive integer. Show that

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1}).$$

(ii) Let a also be a positive integer. Show that if $a^n - 1$ is prime then $a = 2$ and n is prime.

Is it true that if n is prime then $2^n - 1$ is also prime?

4. Let a, b, r, s be rational numbers with $s \neq 0$. Suppose that the number $r + s\sqrt{2}$ is a root of the quadratic equation

$$x^2 + ax + b = 0.$$

Show that $r - s\sqrt{2}$ is also a root.

5. (i) The cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β, γ , and so factorises as

$$a(x - \alpha)(x - \beta)(x - \gamma).$$

Determine

$$\alpha + \beta + \gamma, \quad \alpha\beta + \beta\gamma + \gamma\alpha, \quad \alpha\beta\gamma,$$

in terms of a, b, c, d . What does $\alpha^2 + \beta^2 + \gamma^2$ equal?

(ii) Show that $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

(iii) By considering the roots of the equation $4x^3 - 3x - \cos 3\theta = 0$ deduce that

$$\cos \theta \cos(\theta + 2\pi/3) \cos(\theta + 4\pi/3) = \frac{\cos(3\theta)}{4}.$$

What do

$$\cos \theta + \cos(\theta + 2\pi/3) + \cos(\theta + 4\pi/3) \quad \text{and} \quad \cos^2 \theta + \cos^2(\theta + 2\pi/3) + \cos^2(\theta + 4\pi/3)$$

equal?

ALGEBRA EXERCISES 2

1. Under what conditions on the real numbers a, b, c, d, e, f do the simultaneous equations

$$ax + by = e \quad \text{and} \quad cx + dy = f$$

have (a) a unique solution, (b) no solution, (c) infinitely many solutions in x and y .

Select values of a, b, c, d, e, f for each of these cases, and sketch on separate axes the lines $ax + by = e$ and $cx + dy = f$.

2. For what values of a do the simultaneous equations

$$\begin{aligned} x + 2y + a^2z &= 0, \\ x + ay + z &= 0, \\ x + ay + a^2z &= 0, \end{aligned}$$

have a solution other than $x = y = z = 0$. For each such a find the general solution of the above equations.

3. Do 2×2 matrices exist satisfying the following properties? Either find such matrices or show that no such exist.

- (i) A such that $A^5 = I$ and $A^i \neq I$ for $1 \leq i \leq 4$,
- (ii) A such that $A^n \neq I$ for all positive integers n ,
- (iii) A and B such that $AB \neq BA$,
- (iv) A and B such that AB is invertible and BA is singular (i.e. not invertible),
- (v) A such that $A^5 = I$ and $A^{11} = 0$.

4. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and let} \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

be a 2×2 matrix and its *transpose*. Suppose that $\det A = 1$ and

$$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $a^2 + c^2 = 1$, and hence that a and c can be written as

$$a = \cos \theta \quad \text{and} \quad c = \sin \theta.$$

for some θ in the range $0 \leq \theta < 2\pi$. Deduce that A has the form

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

5. (a) Prove that

$$\det(AB) = \det(A) \det(B)$$

for any 2×2 matrices A and B .

(b) Let A denote the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that

$$A^2 - (\text{trace}A)A + (\det A)I = 0 \tag{1}$$

where

- $\text{trace}A = a + d$ is the trace of A , that is the sum of the diagonal elements;
- $\det A = ad - bc$ is the determinant of A ;
- I is the 2×2 identity matrix.

(c) Suppose now that $A^n = 0$ for some $n \geq 2$. Prove that $\det A = 0$. Deduce using equation (1) that $A^2 = 0$.

CALCULUS EXERCISES 1 — Curve Sketching

1. Sketch the graph of the curve

$$y = \frac{x^2 + 1}{(x - 1)(x - 2)}$$

carefully labelling any turning points and asymptotes.

2. The parabola $x = y^2 + ay + b$ crosses the parabola $y = x^2$ at $(1, 1)$ making right angles.

Calculate the values of a and b .

On the same axes, sketch the two parabolas.

3. The curve C in the xy -plane has equation

$$x^2 + xy + y^2 = 1.$$

By solving $dy/dx = 0$, show that the maximum and minimum values taken by y are $\pm 2/\sqrt{3}$.

By changing to polar co-ordinates, ($x = r \cos \theta$, $y = r \sin \theta$), sketch the curve C .

What is the greatest distance of a point on C from the origin?

4. Sketch the curve $y = x^3 + ax + b$ for a selection of values of a and b .

Suppose now that a is negative. Find the co-ordinates of the turning points of the graph and deduce that $y = 0$ has exactly two roots when

$$b = \pm \frac{2a}{3} \sqrt{\frac{-a}{3}}$$

For what values of b does the equation $y = 0$ have three distinct real roots?

5. On separate xu - and yu -axes sketch the curves $u = 8(x^3 - x)$ and $u = e^y/y$ labelling all turning points.

[Harder] Hence sketch the curve $e^y = 8y(x^3 - x)$.

CALCULUS EXERCISES 2 — Numerical Methods and Estimation

1. Use calculus, or trigonometric identities, to prove the following inequalities for θ in the range $0 < \theta < \frac{\pi}{2}$:
- $\sin \theta < \theta$;
 - $\theta < \tan \theta$;
 - $\cos 2\theta < \cos^2 \theta$.

Hence, without directly calculating the following integrals, rank them in order of size.

$$(a) \int_0^1 x^3 \cos x \, dx, \quad (b) \int_0^1 x^3 \cos^2 x \, dx, \quad (c) \int_0^1 x^2 \sin x \cos x \, dx, \quad (d) \int_0^1 x^3 \cos 2x \, dx.$$

2. Show that the equation

$$\sin x = \frac{1}{2}x$$

has three roots. Using Newton-Raphson, or a similar numerical method, find the positive root to 6 d.p.

The equation $\sin x = \lambda x$ has three real roots when $\lambda = \alpha$ or when $\beta < \lambda < 1$ for two real numbers $\alpha < 0 < \beta$. Plot, on the same axes, the curves

$$y = \sin x, \quad y = \alpha x, \quad y = \beta x.$$

3. Let S denote the circle in the xy -plane with centre $(0,0)$ and radius 1. A regular m -sided polygon I_m is inscribed in S and a regular n -sided polygon C_n is circumscribed about S .

- (a) By considering the perimeter of I_m and the area bounded by C_n , prove that:

$$m \sin \left(\frac{\pi}{m} \right) < \pi < n \tan \left(\frac{\pi}{n} \right),$$

for all natural numbers $m, n \geq 3$.

- (b) Archimedes showed (using this method) that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. What are the smallest values of m and n needed to verify Archimedes' inequality?

4. Find the coefficients of $1, x, x^2, x^3, x^4$ in the power series expansion (Taylor's series expansion) for $f(x) = \sec x$.

Use this approximation to make an estimate for $\sec \frac{1}{10}$. With the aid of a calculator, find to how many decimal places the approximation is accurate.

5. Show that $\int \ln x \, dx = x \ln x - x + \text{constant}$

Sketch the graph of the equation $y = \ln x$. By consideration of areas on your graph, show that

$$n \ln n - n + 1 < \sum_1^n \ln r < (n+1) \ln(n+1) - n.$$

Let $G_n = \sqrt[n]{n!}$ denote the geometric mean of $1, 2, \dots, n$. Show that G_n/n approaches $1/e$ as n becomes large

CALCULUS EXERCISES 3 — Techniques of Integration

1. Evaluate

$$\int \frac{\ln x}{x} dx, \quad \int x \sec^2 x dx, \quad \int_3^\infty \frac{dx}{(x-1)(x-2)}, \quad \int_0^1 \tan^{-1} x dx, \quad \int_0^1 \frac{dx}{e^x + 1}.$$

2. Evaluate, using trigonometric and/or hyperbolic substitutions,

$$\int \frac{dx}{x^2 + 1}, \quad \int_1^2 \frac{dx}{\sqrt{x^2 - 1}}, \quad \int \frac{dx}{\sqrt{4 - x^2}}, \quad \int_2^\infty \frac{dx}{(x^2 - 1)^{3/2}}$$

3. By completing the square in the denominator, and using the substitution

$$x = \frac{\sqrt{2}}{3} \tan \theta - \frac{1}{3}$$

evaluate

$$\int \frac{dx}{3x^2 + 2x + 1}.$$

By similarly completing the square in the following denominators, and making appropriate trigonometric and/or hyperbolic substitutions, evaluate the following integrals

$$\int \frac{dx}{\sqrt{x^2 + 2x + 5}}, \quad \int_0^\infty \frac{dx}{4x^2 + 4x + 5}.$$

4. Let $t = \tan \frac{1}{2}\theta$. Show that

$$\sin \theta = \frac{2t}{1+t^2}, \quad \cos \theta = \frac{1-t^2}{1+t^2}, \quad \tan \theta = \frac{2t}{1-t^2}$$

and that

$$d\theta = \frac{2 dt}{1+t^2}.$$

Use the substitution $t = \tan \frac{1}{2}\theta$ to evaluate

$$\int_0^{\pi/2} \frac{d\theta}{(1 + \sin \theta)^2}.$$

5. Let

$$I_n = \int_0^{\pi/2} x^n \sin x dx.$$

Evaluate I_0 and I_1 .

Show, using integration by parts, that

$$I_n = n \left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}.$$

Hence, evaluate I_5 and I_6 .

CALCULUS EXERCISES 4 — Differential Equations

1. Find the general solutions of the following separable differential equations.

$$\frac{dy}{dx} = \frac{x^2}{y}, \quad \frac{dy}{dx} = \frac{\cos^2 x}{\cos^2 2y}, \quad \frac{dy}{dx} = e^{x+2y}.$$

2. Find the solution of the following initial value problems. On separate axes sketch the solution to each problem.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1-2x}{y}, & y(1) &= -2, \\ \frac{dy}{dx} &= \frac{x(x^2+1)}{4y^3}, & y(0) &= \frac{-1}{\sqrt{2}}, \\ \frac{dy}{dx} &= \frac{1+y^2}{1+x^2} & \text{where } y(0) &= 1. \end{aligned}$$

3. The equation for *Simple Harmonic Motion*, with constant frequency ω , is

$$\frac{d^2x}{dt^2} = -\omega^2x.$$

Show that

$$\frac{d^2x}{dt^2} = v \frac{dv}{dx}$$

where $v = dx/dt$ denotes velocity. Find and solve a separable differential equation in v and x given that $x = a$ when $v = 0$.

Hence show that

$$x(t) = a \sin(\omega t + \varepsilon)$$

for some constant ε .

4. Find the most general solution of the following homogeneous constant coefficient differential equations:

$$\begin{aligned} \frac{d^2y}{dx^2} - y &= 0, \\ \frac{d^2y}{dx^2} + 4y &= 0, \quad \text{where } y(0) = y'(0) = 1, \\ \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y &= 0, \\ \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y &= 0, \quad \text{where } y(0) = y'(0) = 1. \end{aligned}$$

5. Write the left hand side of the differential equation

$$(2x + y) + (x + 2y) \frac{dy}{dx} = 0,$$

in the form

$$\frac{d}{dx}(F(x, y)) = 0,$$

where $F(x, y)$ is a polynomial in x and y . Hence find the general solution of the equation.

Use this method to find the general solution of

$$(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1) \frac{dy}{dx} = 0.$$

CALCULUS EXERCISES 5 — Further Differential Equations

1. Find all solutions of the following separable differential equations:

$$\begin{aligned}\frac{dy}{dx} &= \frac{y - xy}{xy - x}, \\ \frac{dy}{dx} &= \frac{\sin^{-1} x}{y^2 \sqrt{1 - x^2}}, \quad y(0) = 0. \\ \frac{d^2 y}{dx^2} &= (1 + 3x^2) \left(\frac{dy}{dx} \right)^2 \quad \text{where } y(1) = 0 \text{ and } y'(1) = \frac{-1}{2}.\end{aligned}$$

2. Use the method of integrating factors to solve the following equations with initial conditions

$$\begin{aligned}\frac{dy}{dx} + xy &= x \quad \text{where } y(0) = 0, \\ 2x^3 \frac{dy}{dx} - 3x^2 y &= 1 \quad \text{where } y(1) = 0, \\ \frac{dy}{dx} - y \tan x &= 1 \quad \text{where } y(0) = 1.\end{aligned}$$

3. Find the most general solution of the following inhomogeneous constant coefficient differential equations:

$$\begin{aligned}\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y &= x, \\ \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y &= \sin x, \\ \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y &= e^x, \\ \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y &= e^{-x}.\end{aligned}$$

4. (a) By making the substitution $y(x) = xv(x)$ in the following *homogeneous polar equations*, convert them into separable differential equations involving v and x , which you should then solve

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2 + y^2}{xy}, \\ x \frac{dy}{dx} &= y + \sqrt{x^2 + y^2}.\end{aligned}$$

(b) Make substitutions of the form $x = X + a$, $y = Y + b$, to turn the differential equation

$$\frac{dy}{dx} = \frac{x + y - 3}{x - y - 1}$$

into a homogeneous polar differential equation in X and Y . Hence find the general solution of the above equation.

5. A particle P moves in the xy -plane. Its co-ordinates $x(t)$ and $y(t)$ satisfy the equations

$$\frac{dy}{dt} = x + y \quad \text{and} \quad \frac{dx}{dt} = x - y,$$

and at time $t = 0$ the particle is at $(1, 0)$. Find, and solve, a homogeneous polar equation relating x and y .

By changing to polar co-ordinates ($r^2 = x^2 + y^2$, $\tan \theta = y/x$), sketch the particle's journey for $t \geq 0$.

COMPLEX NUMBERS EXERCISES

1. By writing $\omega = a + ib$ (where a and b are real), solve the equation

$$\omega^2 = -5 - 12i.$$

Hence find the two roots of the quadratic equation

$$z^2 - (4 + i)z + (5 + 5i) = 0.$$

2. By substituting $z = x + iy$ or $z = re^{i\theta}$ into the following equations and inequalities, sketch the following regions of the complex plane on separate Argand diagrams:

- $|z - 3 - 4i| < 5$,
- $\arg(z) = \pi/3$
- $0 \leq \operatorname{Re}((iz + 3)/2) < 2$,
- $e^z = 1$,
- $\operatorname{Im}(z^2) < 0$.

3. Find the image of the point $z = 2 + it$ under each of the following transformations.

- $z \mapsto iz$,
- $z \mapsto z^2$,
- $z \mapsto e^z$,
- $z \mapsto 1/z$.

By letting t vary over all real values find the image of the line $\operatorname{Re} z = 2$ under the same transformations.

4. (a) Given that $e^{i\theta} = \cos \theta + i \sin \theta$, prove that

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

(b) Use *De Moivre's Theorem* to show that

$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta.$$

5. (a) Let $z = \cos \theta + i \sin \theta$ and let n be an integer. Show that

$$2 \cos \theta = z + \frac{1}{z} \quad \text{and that} \quad 2i \sin \theta = z - \frac{1}{z}.$$

Find expressions for $\cos n\theta$ and $\sin n\theta$ in terms of z .

(b) Show that

$$\cos^5 \theta = \frac{1}{16} (\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta)$$

and hence find $\int_0^{\pi/2} \cos^5 \theta \, d\theta$.

GEOMETRY EXERCISES

1. Describe the regions of space given by the following vector equations. In each, \mathbf{r} denotes the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; ‘ \cdot ’ and \wedge denote the scalar (dot) and vector (cross) product:

- $\mathbf{r} \wedge (\mathbf{i} + \mathbf{j}) = (\mathbf{i} - \mathbf{j})$,
- $\mathbf{r} \cdot \mathbf{i} = 1$,
- $|\mathbf{r} - \mathbf{i}| = |\mathbf{r} - \mathbf{j}|$,
- $|\mathbf{r} - \mathbf{i}| = 1$,
- $\mathbf{r} \cdot \mathbf{i} = \mathbf{r} \cdot \mathbf{j} = \mathbf{r} \cdot \mathbf{k}$,
- $\mathbf{r} \wedge \mathbf{i} = \mathbf{i}$.

2. Find the shortest distance between the lines

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z}{2} \quad \text{and} \quad x=2, \quad \frac{y-1}{2} = z.$$

[Hint: parametrise the lines and write down the vector between two arbitrary points on the lines; then determine when this vector is normal to both lines.]

3. Let L_θ denote the line through (a, b) making an angle θ with the x -axis. Show that L_θ is a tangent of the parabola $y = x^2$ if and only if

$$\tan^2 \theta - 4a \tan \theta + 4b = 0.$$

[Hint: parametrise L_θ as $x = a + \lambda \cos \theta$ and $y = b + \lambda \sin \theta$ and determine when L_θ meets the parabola precisely once.]

Show that the tangents from (a, b) to the parabola subtend an angle $\pi/4$ if and only if

$$1 + 24b + 16b^2 = 16a^2.$$

[Hint: use the formula $\tan(\theta_1 - \theta_2) = (\tan \theta_1 - \tan \theta_2)/(1 + \tan \theta_1 \tan \theta_2)$.]

Sketch the curve $1 + 24y + 16y^2 = 16x^2$ and the original parabola on the same axes.

4. What transformations of the xy -plane do the following matrices represent:

$$\begin{array}{ll} i) \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, & ii) \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \\ iii) \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, & iv) \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{array}$$

Which, if any, of these transformations are invertible?

5. The *cycloid* is the curve given parametrically by the equations

$$x(t) = t - \sin t, \quad \text{and} \quad y(t) = 1 - \cos t \quad \text{for } 0 \leq t \leq 2\pi.$$

(a) Sketch the cycloid.

(b) Find the arc-length of the cycloid.

(c) Find the area bounded by the cycloid and the x -axis.

(d) Find the area of the surface of revolution generated by rotating the cycloid around the x -axis.

(e) Find the volume enclosed by the surface of revolution generated by rotating the cycloid around the x -axis.