

## PROBLEM SHEET 1

**1.1** Find the radius and centre of the circle described by the equation

$$x^2 + y^2 - 2x - 4y + 1 = 0$$

by writing it in the form  $(x - a)^2 + (y - b)^2 = c^2$  for suitable  $a, b$  and  $c$ .

**1.2** Find the equation of the line perpendicular to  $y = 3x$  passing through the point  $(3, 9)$ .

**1.3** Given

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \quad \text{and} \quad \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

show that

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)] \quad \text{and} \quad \sin^2 A = \frac{1}{2}[1 - \cos 2A].$$

**1.4** Show that

$$4 \cos(\alpha t) + 3 \sin(\alpha t) = 5 \cos(\alpha t + \phi)$$

where  $\phi = \arctan(-3/4)$ .

**1.5** Show that, for  $-1 \leq x \leq 1$ ,

$$\cos(\sin^{-1} x) = \pm \sqrt{1 - x^2}.$$

**1.6** Given

$$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B \quad \text{and} \quad \cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B,$$

show that

$$\cosh A \cosh B = \frac{1}{2}[\cosh(A + B) + \cosh(A - B)] \quad \text{and} \quad \sinh^2 A = \frac{1}{2}[\cosh 2A - 1].$$

**1.7** Given that

$$\sinh x = \frac{1}{2}[e^x - e^{-x}],$$

show that

$$\sinh^{-1} x = \ln \left[ x + \sqrt{1 + x^2} \right].$$

**1.8** Express

$$\frac{x}{(x - 1)(x - 2)}$$

in partial fractions.

**1.9** If  $a_n = \frac{1}{n}$ , find  $\sum_{i=1}^5 a_n$  as a fraction.

**1.10** If  $S = \sum_{i=0}^N x^i$ , show that  $xS = \sum_{i=1}^{N+1} x^i$ . Hence show that  $S - xS = 1 - x^{N+1}$  and therefore that

$$S = \frac{1 - x^{N+1}}{1 - x}.$$

## PROBLEM SHEET 2

**2.1** Given that

$$\sinh x = \frac{1}{2}[e^x - e^{-x}]$$

show that

$$\frac{dy}{dx} = \cosh x.$$

**2.2** Given that

$$\cosh x = \frac{1}{2}[e^x + e^{-x}],$$

show that

$$\frac{dy}{dx} = \sinh x.$$

**2.3** Let  $n$  be a positive integer. Show that

$$\frac{d^n(x^n)}{dx^n} = n!$$

**2.4** If  $y = \ln x$ , show that

$$\frac{dy}{dx} = \frac{1}{x}; \quad \frac{d^2y}{dx^2} = \frac{-1}{x^2}; \quad \frac{d^{100}y}{dx^{100}} = \frac{-99!}{x^{100}}.$$

**2.5** Find the equation of the tangent to the curve  $y = x^2$  at  $(1, 1)$ .

**2.6** Find the slope of the curve  $y = 4x + e^x$  at  $(0, 1)$ .

**2.7** Find the angle of inclination of the tangent to the curve  $y = x^2 + x + 1$  at the point  $(0, 1)$ .

**2.8** The displacement  $y(t)$  metres of a body at time  $t$  seconds ( $t \geq 0$ ) is given by  $y(t) = t - \sin t$ . At what times is the body at rest?

**2.9** A particle has displacement  $y(t)$  metres at time  $t$  seconds given by  $y(t) = 3t^3 + 4t + 1$ . Find its acceleration at time  $t = 4$  seconds.

**2.10** If

$$y = \sum_{n=0}^N a_n x^n$$

show that

$$\frac{dy}{dx} = \sum_{n=1}^N n a_n x^{n-1}.$$

### PROBLEM SHEET 3

**3.1** If  $y = \ln(1 + x^2)$ , find  $dy/dx$ .

**3.2** If

$$y = \frac{x}{1 + x^2}$$

find  $dy/dx$ .

**3.3** If  $y = \cosh(x^4)$ , find  $dy/dx$ .

**3.4** If  $y = x^2 \ln x$ , find  $d^2y/dx^2$ .

**3.5** Find  $dy/dx$  for  $y = (1 + x^2)^{-1/2}$ .

**3.6** Show that for  $y = \sinh^{-1} x$ ,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}.$$

**3.7** Show that for  $y = \ln[x + \sqrt{1 + x^2}]$ ,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}.$$

**3.8** Find  $dy/dx$  for  $y = \cos^{-1}(\sin x)$ .

**3.9** A curve is given in polar coordinates by  $r = 1 + \sin^2\theta$ . Find  $dy/dx$  at  $\theta = \pi/4$ .

**3.10** Show that if

$$y = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right|, \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{x^2 - a^2}.$$

## PROBLEM SHEET 4

**4.1** Given  $f(x - ct)$ , where  $x$  and  $c$  are constant, show that

$$\frac{d^2}{dt^2}f(x - ct) = c^2 f''(x - ct),$$

and calculate this expression when  $f(u) = \sin u$ .

**4.2** Classify the stationary point of  $y = x^{-2}\ln x$ , where  $x > 0$ .

**4.3** Classify the stationary points of  $y(x) = x^2 - 3x + 2$ .

**4.4** The numbers  $x$  and  $y$  are subject to the constraint  $x + y = \pi$ . Find the values of  $x$  and  $y$  for which  $\cos(x)\sin(y)$  takes its minimum value.

**4.5** Sketch the graph of

$$y = \frac{x}{1 + x^2}.$$

**4.6** Sketch the graph of

$$y(x) = \tan(2x) \quad \text{for} \quad -\frac{3\pi}{4} \leq x \leq \frac{3\pi}{4}.$$

**4.7** Sketch the graph of  $y = x \ln x$  for  $x > 0$ .

**4.8** Sketch the graph of

$$y = \frac{x^3}{2x - 1}$$

showing clearly on your sketch any asymptotes.

**4.9** Sketch the graph of

$$y = x \cos(3x) \quad \text{for} \quad 0 \leq x \leq 2\pi.$$

## PROBLEM SHEET 5

**5.1** Verify the following Taylor expansions (taking the ranges of validity for granted).

(a)

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots \text{valid for any } x.$$

(b)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots \text{valid for any } x.$$

(c)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots \text{valid for any } x.$$

(d) Let  $\alpha$  be a constant.

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \text{valid for } -1 < x < 1$$

(e)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n-1}x^n}{n} + \dots \text{valid for } -1 < x \leq 1.$$

**5.2** Obtain a four-term Taylor polynomial approximation valid near  $x = 0$  for each of the following.

$$(a) (1+x)^{1/2}, \quad (b) \sin(2x), \quad (c) \ln(1+3x).$$

## PROBLEM SHEET 6

**6.1** Reduce to standard form

$$(a) \frac{3+i}{4-i}, \quad \text{and} \quad (b) (1+i)^5.$$

**6.2** Prove

$$(a) |z_1 z_2| = |z_1| |z_2|, \quad \text{and} \quad (b) \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad \text{when } z_2 \neq 0.$$

**6.3** Given that  $e^{i\theta} = \cos \theta + i \sin \theta$ , prove that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

**6.4** Let  $z = 1+i$ . Find the following complex numbers in standard form and plot their corresponding points in the Argand diagram:-

$$(a) \bar{z}^2, \quad \text{and} \quad (b) \frac{z}{\bar{z}}.$$

**6.5** Find the modulus and principal arguments of (a)  $-2+2i$ , (b)  $3+4i$ .

**6.6** Find all the complex roots of

- (a)  $\cosh z = 1$ ;
- (b)  $\sinh z = 1$ ;
- (c)  $e^z = -1$ ;
- (d)  $\cos z = \sqrt{2}$ .

**6.7** Show that the mapping

$$w = z + \frac{c}{z},$$

where  $z = x + iy$ ,  $w = u + iv$  and  $c$  is a real number, maps the circle  $|z| = 1$  in the  $z$  plane into an ellipse in the  $w$  plane and find its equation.

**6.8** Show that

$$\cos^6 \theta = \frac{1}{32}(\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10).$$

## PROBLEM SHEET 7

**7.1** The matrix  $A = (a_{ij})$  is given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & -2 & 4 \\ 1 & 5 & -3 \end{pmatrix}$$

Identify the elements  $a_{13}$  and  $a_{31}$ .

**7.2** Given that

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ -1 & 1 \\ -0 & 1 \end{pmatrix},$$

verify the distributive law  $A(B + C) = AB + AC$  for the three matrices.

**7.3** Let

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}.$$

Show that  $AB = 0$ , but that  $BA \neq 0$ .

**7.4** A general  $n \times n$  matrix is given by  $A = (a_{ij})$ . Show that  $A + A^T$  is a symmetric matrix, and that  $A - A^T$  is skew-symmetric.

Express the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

as the sum of a symmetric matrix and a skew-symmetric matrix.

**7.5** Let the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 1 \end{pmatrix}.$$

Find  $A^2$ . For what relation between  $a, b$ , and  $c$  is  $A^2 = I$  (the unit matrix)? In this case, what is the inverse matrix of  $A$ ? What is the inverse matrix of  $A^{2n-1}$  ( $n$  a positive integer)?

**7.6** Using the rule for inverses of  $2 \times 2$  matrices, write down the inverse of

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

**7.7** If  $A$  and  $B$  are both  $n \times n$  matrices with  $A$  non-singular, show that

$$(A^{-1}BA)^2 = A^{-1}B^2A.$$

## PROBLEM SHEET 8

**8.1** Obtain the components of the vectors below where  $L$  is the magnitude and  $\theta$  the angle made with the positive direction of the  $x$  axis ( $-180^\circ < \theta \leq 180^\circ$ ).

- (a)  $L = 3, \theta = 60^\circ$ ;
- (b)  $L = 3, \theta = -150^\circ$ .

**8.2** Two ships,  $S_1$  and  $S_2$  set off from the same point  $Q$ . Each follows a route given by successive displacement vectors. In axes pointing east and north,  $S_1$  follows the path to  $B$  via  $\overrightarrow{QA} = (2, 4)$ , and  $\overrightarrow{AB} = (4, 1)$ .  $S_2$  goes to  $E$  via  $\overrightarrow{QC} = (3, 3)$ ,  $\overrightarrow{CD} = (1, 1)$  and  $\overrightarrow{DE} = (2, -3)$ . Find the displacement vector  $\overrightarrow{BE}$  in component form.

**8.3** Sketch a diagram to show that if  $A, B, C$  are any three points, then  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$ . Formulate a similar result for any number of points.

**8.4** Sketch a diagram to show that if  $A, B, C, D$  are any four points, then  $\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AD}$ . Formulate a similar result for any number of points.

**8.5** Two points  $A$  and  $B$  have position vectors  $\mathbf{a}$  and  $\mathbf{b}$  respectively. In terms of  $\mathbf{a}$  and  $\mathbf{b}$  find the position vectors of the following points on the straight line passing through  $A$  and  $B$ .

- (a) The mid-point  $C$  of  $AB$ ;
- (b) a point  $U$  between  $A$  and  $B$  for which  $AU/UB = 1/3$ .

**8.6** Suppose that  $C$  has position vector  $\mathbf{r}$  and  $\mathbf{r} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$  where  $\lambda$  is a parameter, and  $A, B$  are points with  $\mathbf{a}, \mathbf{b}$  as position vectors. Show that  $C$  describes a straight line. Indicate on a diagram the relative positions of  $A, B, C$ , when  $\lambda < 0, 0 < \lambda < 1$ , and  $\lambda > 1$ .

**8.7** Find the shortest distance from the origin of the line given in vector parametric form by  $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ , where

$$\mathbf{a} = (1, 2, 3) \quad \text{and} \quad \mathbf{b} = (1, 1, 1),$$

and  $t$  is the parameter (Hint: use a calculus method, with  $t$  as the independent variable.)

**8.8**  $ABCD$  is any quadrilateral in three dimensions. Prove that if  $P, Q, R, S$  are the mid-points of  $AB, BC, CD, DA$  respectively, then  $PQRS$  is a parallelogram.

**8.9**  $ABC$  is a triangle, and  $P, Q, R$  are the mid-points of the respective sides  $BC, CA, AB$ . Prove that the medians  $AP, BQ, CR$  meet at a single point  $G$  (called the centroid of  $ABC$ ; it is the centre of mass of a uniform triangular plate.)

**8.10** Show that the vectors  $\overrightarrow{OA} = (1, 1, 2), \overrightarrow{OB} = (1, 1, 1)$ , and  $\overrightarrow{OC} = (5, 5, 7)$  all lie in one plane.



## PROBLEM SHEET 9

**9.1** The figure  $ABCD$  has vertices at  $(0, 0)$ ,  $(2, 0)$ ,  $(3, 1)$  and  $(1, 1)$ .

Find the vectors  $\overrightarrow{AC}$  and  $\overrightarrow{BD}$ . Find  $\overrightarrow{AC} \cdot \overrightarrow{BD}$ .

Hence show that the angles between the diagonals of  $ABCD$  have cosine  $-1/\sqrt{5}$ .

**9.2** Show that the vectors  $\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$  and  $\mathbf{b} = -2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$  are perpendicular.

Obtain any vector  $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$  which is perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ .

**9.3** Find the value of  $\lambda$  such that the vectors  $(\lambda, 2, -1)$  and  $(1, 1, -3\lambda)$  are perpendicular.

**9.4** Find a constant vector parallel to the line given parametrically by

$$x = 1 - \lambda, y = 2 + 3\lambda, z = 1 + \lambda.$$

**9.5** A circular cone has its vertex at the origin and its axis in the direction of the unit vector  $\hat{\mathbf{a}}$ . The half-angle at the vertex is  $\alpha$ . Show that the position vector  $\mathbf{r}$  of a general point on its surface satisfies the equation

$$\hat{\mathbf{a}} \cdot \mathbf{r} = |\mathbf{r}| \cos \alpha.$$

Obtain the cartesian equation when  $\hat{\mathbf{a}} = (2/7, -3/7, -6/7)$  and  $\alpha = 60^\circ$ .

## PROBLEM SHEET 10

**10.1** For vectors  $\mathbf{a}$  and  $\mathbf{b}$ , show

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \quad \text{and} \quad \mathbf{a} \cdot \mathbf{b} = \frac{1}{4}(|\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2)$$

**10.2** In component form, let  $\mathbf{a} = (1, -2, 2)$ ,  $\mathbf{b} = (3, -1, -1)$ , and  $\mathbf{c} = (-1, 0, -1)$ . Evaluate

$$\mathbf{a} \times \mathbf{b}, \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \quad \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

**10.3** What is the geometrical significance of  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ ?

**10.4** Show that the vectors  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$  and  $\mathbf{b} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$  are perpendicular. Find a vector which is perpendicular to  $\mathbf{a}$  and  $\mathbf{b}$ .

**10.5** Let  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  be three non-coplanar vectors, and  $\mathbf{v}$  be any vector. Show that  $\mathbf{v}$  can be expressed as

$$\mathbf{v} = X\mathbf{a} + Y\mathbf{b} + Z\mathbf{c}$$

where  $X, Y, Z$ , are constants given by

$$X = \frac{\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \quad Y = \frac{\mathbf{v} \cdot (\mathbf{c} \times \mathbf{a})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \quad Z = \frac{\mathbf{v} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}.$$

(Hint: start by forming, say,  $\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c})$ ).

## PROBLEM SHEET 11

**11.1** Integrate  $\cos(3x + 4)$ .

**11.2** Integrate  $(1 - 2x)^{10}$ .

**11.3** Integrate  $e^{4x-1}$ .

**11.4** Integrate  $(4x + 3)^{-1}$ .

**11.5** Find the equation of the curve passing through the point  $(1, 2)$  satisfying  $dy/dx = 2x$ .

**11.6** A particle has acceleration  $(3t^2 + 4) \text{ ms}^{-2}$  at time  $t$  seconds. If its initial speed is  $5\text{ms}^{-1}$ , what is its speed at time  $t = 2$  seconds?

**11.7** Find the area between the graph of  $y = \sin x$  and the  $x$ -axis from  $x = 0$  to  $x = \pi/2$ .

**11.8** Find the area between the graph

$$y = \frac{1}{x-1}$$

and the  $x$ -axis between  $x = 2$  and  $x = 3$ .

**11.9** Find the signed area between the graph  $y = 2x + 1$  and the  $x$ -axis between  $x = -1$  and  $x = 3$ .

**11.10** Find  $y$ , given that

$$\frac{d^2y}{dx^2} = \sin x - \frac{4}{x^3}.$$