Patterns and Invariants in Mathematics

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Outline

Pinecones

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$$F_n = F_{n-1} + F_{n-2}$$

The Fibonacci sequence is the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

and occurs in the patterns of spirals we see in pinecones, sunflowers, leaf patterns, etc.

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Invariants allow us to distinguish between two objects or to deduce that an object has certain properties.

Representation theory of algebras

A representation (or module) of an algebra is a way of describing the effect of the action of the algebra. For example, we may consider the way in which a set of rotations acts on objects in three-dimensional space.

Our aim is to study invariants of algebras and their modules through cohomology theories. These enable us to construct invariants (which may be algebraic or numerical), and link mathematical ideas and results in algebra, geometry and topology.

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My research focuses on Ext and on Hochschild cohomology. Knowledge of the cohomology of an algebra allows us to give structural information on all representations (global property) and to describe invariants of individual representations (local property).

Hochschild cohomology

Given an algebra $\mathcal{A}$, the Hochschild cohomology ring of an algebra is denoted by $\mathrm{HH}^*(\mathcal{A})$.

This is a graded algebra

$$\mathrm{HH}^*(\mathcal{A}) = \mathrm{HH}^0(\mathcal{A}) \oplus \mathrm{HH}^1(\mathcal{A}) \oplus \mathrm{HH}^2(\mathcal{A}) \oplus \cdots$$

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The whole ring is usually hard to calculate and understand!

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- When is the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(\mathcal{A})/\mathcal{N}$ finitely generated?
- What information does this give us about the representations of $\mathcal{A}$?

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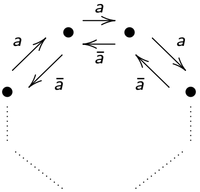
An example

Let $m \geq 1$. Let $\mathcal{A}_m$ be given by the following quiver with m vertices:



and relations: $a^2 = 0$, $\bar{a}^2 = 0$, $a\bar{a} = \bar{a}a$.

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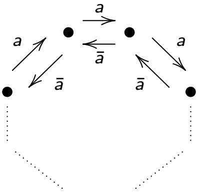

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Then $\mathrm{HH}^*(\mathcal{A}_m)$ is a finitely generated algebra, and hence $\mathrm{HH}^*(\mathcal{A}_m)/\mathcal{N}$ is also finitely generated.

... example continued

Other properties of the algebras $\mathcal{A}_m$ are:

1. If $m = 1$ then $\mathcal{A}_m$ is commutative;
2. If $m \geqslant 2$ then $\mathcal{A}_m$ is not commutative;
3. If $m = 1$ then $\mathrm{HH}^*(\mathcal{A}_m)/\mathcal{N}$ has 2 generators;
4. If $m \geqslant 2$ then $\mathrm{HH}^*(\mathcal{A}_m)/\mathcal{N}$ has 3 generators and is of Krull dimension 2; this measures how close it is to being the set of all polynomials in x, y and z.

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Moreover, the Ext algebra of $\mathcal{A}_m$ is particularly nice!

The Ext algebra and Koszul algebras

The Ext algebra gives another cohomology theory and is related to Hochschild cohomology.

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Question

What happens if we look for algebras where the Ext algebra $E(\mathcal{A})$ is generated in degrees 0, 1 and 2? or in degrees 0, 1, 2 and 3? or ...

Counting

To find the Ext algebra of any algebra $\mathcal{A}$, we need to construct a certain collection of vector spaces $P^0, P^1, P^2, \ldots$ and invariants associated to each of these vector spaces. In the case where $\mathcal{A}$ is a Koszul algebra, then P^n has a single invariant which is n.

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If we restrict ourselves to monomial algebras, then this is it!

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2. The algebras we have considered occur in many different areas of mathematics.
3. Their representation theory is being applied in statistical mechanics, chaos theory and dynamical systems, signal processing and efficient mobile phone communication.